

Computing isolated orbifolds in weighted flag varieties

Muhammad Imran Qureshi

Abstract

Given a weighted flag variety $w\Sigma(\mu, u)$ corresponding to chosen fixed parameters μ and u , we present an algorithm to compute lists of all possible projectively Gorenstein n -folds, having canonical weight k and isolated orbifold points, appearing as weighted complete intersections in $w\Sigma(\mu, u)$ or some projective cone(s) over $w\Sigma(\mu, u)$. We apply our algorithm to compute lists of interesting classes of polarized 3-folds with isolated orbifold points in the codimension 8 weighted G_2 variety. We also show the existence of some families of log-terminal \mathbb{Q} -Fano 3-folds in codimension 8 by explicitly constructing them as quasilinear sections of a weighted G_2 -variety.

1 Introduction

This article has two main parts. In the first part, for a weighted flag variety $w\Sigma(\mu, u)$ corresponding to the fixed chosen parameters μ and u , we give an algorithm to compute all possible families of isolated orbifolds of fixed dimension n and canonical weight k that are complete intersections in $w\Sigma(\mu, u)$ or some projective cone(s) over $w\Sigma(\mu, u)$. The dimension n of isolated orbifolds is required to be greater than or equal to 2, due to Theorem 2.7, and k is an integer such that the canonical divisor class $K_X \sim kD$, for an ample \mathbb{Q} -Cartier divisor D . In the second part, we use the algorithm to compute numerical candidate lists of different types of families of isolated 3-dimensional orbifolds whose general member is a quasilinear section of a certain weighted flag variety $w\Sigma(\mu, u)$, also called format. In particular, we obtain lists of canonical 3-folds, Calabi–Yau 3-folds and log-terminal \mathbb{Q} -Fano 3-folds. The lists are obtained by implementing the algorithm in the computer algebra system MAGMA [BCP97].

We are interested in computing lists of well-formed and quasi-smooth projectively Gorenstein polarized n -folds (X, D) , where X is polarized by an ample \mathbb{Q} -Cartier divisor D , appearing as quasilinear sections of some fixed weighted flag variety $w\Sigma(\mu, u)$. The algorithm to compute such lists of candidate families of n -folds is primarily based on the Hilbert series formula of Buckley–Reid–Zhou [BRZ13, Thrm. 1.3]. The formula gives the decomposition of the Hilbert series $P_X(t) = P_I(t) + P_Q(t)$: $P_I(t)$ represents the smooth part and $P_Q(t)$ represents the orbifold part of the n -fold (X, D) . The term $P_Q(t)$ will have the form $\sum_{Q_i \in \mathcal{B}} P_{Q_i}(t)$, where \mathcal{B} is a collection of singularities (possibly with repeats) known as basket, coming from the embedding of X in some weighted projective space.

The algorithm starts with computing the Hilbert series and the canonical divisor class $\mathcal{O}_{w\Sigma}(-p)$ of some weighted flag variety $w\Sigma(\mu, u)$ of dimension d , where $p \gg 0$ is a positive integer. The flag varieties are known to be Fano varieties, whence $-p$ for the canonical weight of $w\Sigma$. Then we find all possible n -fold quasilinear sections X of $w\Sigma$ or of projective cone(s) over $w\Sigma$ such that $\mathcal{O}_X(K_X) = \mathcal{O}_X(kD)$. For a chosen n -fold model X , we find its Hilbert series $P_X(t)$

and the corresponding initial term $P_I(t)$. Since we are interested in quasi-smooth orbifolds, the singularities of X come from the singularities of the weighted projective space containing X . We compute all the possible baskets \mathcal{B} of isolated singularities and their contributions $P_{Q_i}(t)$ to the Hilbert series $P_X(t)$. In the last step we run through each basket to determine the multiplicities of the orbifold terms $P_{Q_i}(t)$: the values of m_i in the equation

$$P_X(t) = P_I(t) + \sum m_i P_{Q_i}(t). \quad (1.1)$$

Then X is a suitable candidate for an isolated polarized n -fold if $m_i \geq 0$ for all i . In fact, the algorithm can be used to find lists of isolated n -folds which may be realized as weighted complete intersection in any ambient weighted projective variety having computable canonical divisor class and Hilbert series.

We implement the algorithm in the computer algebra system MAGMA to compute lists of candidate families of isolated 3-folds in two cases: the codimension 8 weighted G_2 variety and the codimension 3 weighted $\text{Gr}(2, 5)$. We compute the numerical candidate families of log-terminal \mathbb{Q} -Fano 3-folds, Calabi–Yau 3-folds and canonical 3-folds whose general member is a weighted complete intersection in the corresponding ambient variety. The lists are computed by ordering the input parameters (μ, u) in order of increasing the sum of the weights W of the embedding containing $w\Sigma$. The lists do not present the full classification of such 3-folds in a chosen weighted flag variety $w\Sigma$ but are complete up to a certain value of W in each case. The list of 33 candidate log-terminal \mathbb{Q} -Fano 3-folds has been explicitly checked by using their defining equations; 6 of them exist as actual polarized 3-folds with the desired properties, given in Theorem 4.1. The list also confirms the non-existence of terminal \mathbb{Q} -Fano 3-folds in the weighted G_2 variety. The lists of families of Calabi–Yau 3-folds and canonical 3-folds shall only be considered as numerical candidate 3-folds in the G_2 case. Their existence can be checked by using their defining equations under their Plücker-style embeddings, and checking their type of singularities using those equations. The case $w \text{Gr}(2, 5)$ has been included to check the algorithm against already existing lists of polarized 3-folds.

The main motivation to constructing polarized varieties as complete intersections in weighted flag varieties $w\Sigma$ comes from Mukai’s linear section theorem [Muk88, Muk89]: every prime Gorenstein Fano 3-fold of genus $7 \leq g \leq 10$ is a linear section of some flag variety. The idea was first generalized to the weighted case by Corti–Reid in [CR02] to construct some 3-folds and surfaces with quotient singularities in codimension 3 and 5. They recovered the list of 69 terminal \mathbb{Q} -Fano 3-folds of Altmok as a quasilinear section of $w \text{Gr}(2, 5)$ or of a projective cone over it. The terminal \mathbb{Q} -Fano 3-folds form a bounded family of varieties and more importantly lie in the Mori category of varieties: they are minimal models in dimension 3. Their existence in codimension 1–4 has been established in the past, for instance see [ABR02, BKR12]. Our algorithmic approach allows one to search for terminal \mathbb{Q} -Fano 3-folds in higher codimensions. The previous attempts [QS11, Qur15] of constructing them as quasilinear sections of some weighted flag varieties were not successful but using our algorithmic approach, one can at least confirm the non-existence of these varieties in the corresponding weighted flag varieties. Since every quotient singularity is log-terminal [Kaw84], the \mathbb{Q} -Fano 3-folds computed in section 4 are log-terminal. The class of log-terminal \mathbb{Q} -Fano 3-folds also forms a bounded family of algebraic varieties [Bor96]. We construct some families of isolated log-terminal \mathbb{Q} -Fano 3-folds as weighted complete intersection in the codimension 8 weighted G_2 variety.

Acknowledgements

I am grateful to Gavin Brown for some stimulating discussions which made me think about this project. I am also thankful to Sohail Iqbal, Balázs Szendrői and Shengtian Zhou for fruitful discussions and Miles Reid for making available his MAGMA function `Qorb`. Thanks are also due to Klaus Altmann and Berlin Mathematical school for their hospitality during a part of this project at the Freie Universität Berlin. Last but not least, I wish to thank an anonymous referee whose comments on an earlier version helped me a great deal to improve the presentation of the article. This research is supported by the LUMS's faculty start up research grant.

2 Definitions and tools

2.1 Baskets and polarized orbifolds

Let the multiplicative group of r -th roots of unity μ_r acts on \mathbb{A}^n by diagonal representation

$$\mu_r \ni \varepsilon : (x_1, \dots, x_n) \mapsto (\varepsilon^{a_1} x_1, \dots, \varepsilon^{a_n} x_n).$$

Then the quotient $\pi : \mathbb{A}^n \rightarrow \mathbb{A}/\mu_r$ is called a cyclic quotient singularity of type $\frac{1}{r}(a_1, \dots, a_n)$. The cyclic quotient singularity is called isolated if $\gcd(r, a_i) = 1$ for all $1 \leq i \leq n$. We call a collection (with possible repeats) of singularities $\frac{1}{r_j}(a_{1_j}, \dots, a_{n_j})$ the basket of singularities, denoted by \mathcal{B} .

A polarized n -fold is a pair (X, D) , where X is an n -dimensional projective algebraic variety and D is a \mathbb{Q} -ample Weil divisor on X . All our polarized n -folds are well-formed and quasi-smooth; appearing as projective subvarieties of some weighted projective space denoted by $\mathbb{P}[w_0, \dots, w_N]$ or $w\mathbb{P}^N$ or $\mathbb{P}[w_i]$. We call a weighted projective variety $X \subset \mathbb{P}[w_i]$ of codimension e well-formed if the singular locus of X does not contain the codimension $e+1$ singular strata of $\mathbb{P}[w_i]$. The subvariety $X \subset \mathbb{P}[w_i]$ is quasi-smooth if the affine cone $\tilde{X} = \text{Spec } R(X, D) \subset \mathbb{A}^{N+1}$ of X is smooth outside its vertex $\underline{0}$. Thus all our polarized n -folds (X, D) are orbifolds, i.e. they only have quotient singularities induced by the singular strata of $\mathbb{P}[w_i]$. In particular, we are interested in orbifolds with only isolated cyclic quotient singularities. A well-formed n -fold is called projectively Gorenstein if

1. $H^i(X, \mathcal{O}_X(m)) = 0$ for all m and $0 < i < n$;
2. the orbifold canonical sheaf of X is given by $\omega_X \sim \mathcal{O}_X(k)$.

The integer k is known as the canonical weight of the n -fold X . A variety X is said to have terminal (log-terminal) singularities if, given a resolution of singularities $f : Y \rightarrow X$, with

$$K_Y = f^*(K_X) + \sum a_i E_i, \quad E_i \text{ are exceptional divisors and } a_i \in \mathbb{Q},$$

we have $a_i > 0$ ($a_i > -1$).

2.2 Graded rings and Hilbert Series

Given a polarized n -fold (X, D) , the associated finite dimensional vector spaces $H^0(X, \mathcal{O}_X(mD))$ fit together to give rise to a finitely generated graded ring

$$R(X, D) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD)),$$

with

$$X \cong \text{Proj } R(X, D) \hookrightarrow \mathbb{P}[w_0, \dots, w_N].$$

The Hilbert series of a polarized projective variety (X, D) , which is the Hilbert series of the graded ring $R(X, D)$, is given by

$$P_{(X, D)}(t) = \sum_{m \geq 0} h^0(X, mD) t^m,$$

where $h^0(X, mD) = \dim H^0(X, \mathcal{O}_X(mD))$. We usually write $P_X(t)$ for the Hilbert series for the sake of brevity. By standard Hilbert–Serre theorem [AM69, Theorem 11.1], $P_X(t)$ has a compact form

$$P_X(t) = \frac{H(t)}{\prod_{i=0}^N (1 - t^{w_i})}. \quad (2.1)$$

The Hilbert numerator $H(t)$ is a Gorenstein symmetric polynomial of degree q : $t^q H(1/t) = (-1)^e H(t)$ where e is the codimension of X . The polynomial $H(t)$ has the form $1 - \sum t^{b_{0j}} + \sum t^{b_{1j}} - \dots + (-1)^c t^q$, where b_{0j} are the degrees of the equations, b_{1j} the degrees of the first syzygies, and so on. The degree q of $H(t)$ is called the adjunction number of X .

2.3 Weighted flag varieties

Let G be a reductive Lie group, with fixed Borel and maximal torus $T \subset B \subset G$. Let $\Lambda_W = \text{Hom}(T, \mathbb{C}^*)$ be the weight lattice of G and let V_λ denote the G -representation with highest weight λ . Then there is an embedding $\Sigma \hookrightarrow \mathbb{P}V_\lambda$ of a flag variety $\Sigma = G/P$, where $P = P_\lambda$ is the parabolic subgroup of G corresponding to the set of simple roots of G orthogonal to the weight vector λ .

Let $\Lambda_W^* = \text{Hom}(\mathbb{C}^*, T)$ be the lattice of one-parameter subgroups of G . Choose $\mu \in \Lambda_W^*$ and a non-negative integer $u \in \mathbb{Z}$ such that $\langle w\lambda, \mu \rangle + u > 0$ for all elements w of the Weyl group W of the Lie group G , where $\langle \cdot, \cdot \rangle$ denotes the perfect pairing between Λ_W and Λ_W^* . Then we define the weighted flag variety $w\Sigma \subset w\mathbb{P}V_\lambda$ following Corti–Reid [CR02]: take the affine cone $\tilde{\Sigma} \subset V_\lambda$ of the embedding $\Sigma \subset \mathbb{P}V_\lambda$ and quotient out by the \mathbb{C}^* -action on $V_\lambda \setminus \{0\}$ defined by

$$(\varepsilon \in \mathbb{C}^*) \mapsto (v \mapsto \varepsilon^u (\mu(\varepsilon) \circ v)).$$

The notation $w\Sigma$ will refer to general weighted flag variety and $w\Sigma(\mu, u)$ for the weighted flag variety with chosen fixed parameters μ and u . We use the term format for each $w\Sigma(\mu, u)$ following [BKZ].

Theorem 2.4 [QS11, Thm. 3.1] The Hilbert series of the weighted flag variety $(w\Sigma(\mu, u), D)$ has the following closed form.

$$P_{w\Sigma}(t) = \frac{\sum_{w \in W} (-1)^w \frac{t^{\langle w\rho, \mu \rangle}}{(1 - t^{\langle w\lambda, \mu \rangle + u})}}{\sum_{w \in W} (-1)^w t^{\langle w\rho, \mu \rangle}}. \quad (2.2)$$

Here ρ is the Weyl vector, half the sum of the positive roots of G , and $(-1)^w = 1$ or -1 depending on whether w consists of an even or odd number of simple reflections in the Weyl group W ; and D is obtained as the pullback of $\mathcal{O}_{w\Sigma}(1)$ under the embedding $w\Sigma \subset w\mathbb{P}V_\lambda$.

Remark 2.5 The Hilbert series expression (2.2) always reduces to the standard expression (2.1) after performing some simplifications, see [QS11, QS12] for details.

2.6 Hilbert series of isolated orbifolds

The integral part of our algorithm is based on the following theorem of Buckley–Reid–Zhou [BRZ13]. The theorem gives the decomposition of the Hilbert series of an isolated orbifold (X, D) as a sum of two expressions: the initial term $P_I(t)$ which represents the smooth part of $P_X(t)$ and the orbifold term $\sum_{Q_i \in \mathcal{B}} P_{Q_i}(t)$ which represents the singular part of $P_X(t)$.

Theorem 2.7 [BRZ13] Let (X, D) be a projectively Gorenstein quasi-smooth orbifold of dimension $n \geq 2$ and canonical weight k , with only isolated orbifold points, given by

$$\mathcal{B} = \{Q_i \text{ of type } \frac{1}{r_i}(a_1, \dots, a_n)\}$$

as its only singularities. Then the Hilbert series $P_X(t) = \sum_{m \geq 0} h^0(X, \mathcal{O}_X(mD))t^m$ of X has the form

$$P_X(t) = P_I(t) + \sum_{Q_i \in \mathcal{B}} P_{Q_i}(t), \quad (2.3)$$

with initial term $P_I(t)$ and each orbifold term $P_{Q_i}(t)$ characterised as follows:

- The initial term

$$P_I(t) = \frac{A(t)}{(1-t)^{n+1}} \quad (2.4)$$

has numerator $A(t)$, an integral Gorenstein symmetric polynomial, of degree equal to the coindex $c = k + n + 1$ of X , so that $P_I(t)$ equals $P_X(t)$ up to degree $\lfloor \frac{c}{2} \rfloor$ and $P_I = 0$ for $c < 0$.

- Each orbifold point $Q_i \in \mathcal{B}$ contributes

$$P_{Q_i}(t) = \frac{B(t)}{(1-t)^n(1-t^r)} \quad (2.5)$$

to the Hilbert series where the numerator

$$B(t) = \text{InvMod} \left(\prod_{i=1}^n \frac{1-t^{a_i}}{1-t}, \frac{1-t^r}{1-t}, \left\lfloor \frac{c}{2} \right\rfloor + 1 \right).$$

This is the unique integral Laurent polynomial supported in $\left[\left\lfloor \frac{c}{2} \right\rfloor + 1, \left\lfloor \frac{c}{2} \right\rfloor + r - 1 \right]$ equal to the inverse of $\prod_{i=1}^n \frac{1-t^{a_i}}{1-t}$ modulo $\frac{1-t^r}{1-t}$. The numerator $B(t)$ is a Gorenstein symmetric polynomial of degree $k + n + r$.

Example 2.8 Consider the Hilbert series of the canonical 3-fold hypersurface of degree 7:

$$X_7 \hookrightarrow \mathbb{P}[1, 1, 1, 1, 2].$$

X_7 contains an isolated singular point of type $\mathcal{B} = \{\frac{1}{2}(1, 1, 1)\}$. The Hilbert series of X_7 is given by

$$P_X(t) = \frac{1 - t^7}{(1 - t)^4(1 - t^2)}.$$

The coindex of X_7 is given by $c = 1 + 1 + 3 = 5$. The Hilbert series has a decomposition into the smooth part $P_I(t)$ and the orbifold part $P_Q(t)$. The smooth part is given by

$$P_I(t) = \frac{1 + t^2 + t^3 + t^5}{(1 - t)^4}$$

and the orbifold part is given by

$$P_Q(t) = \frac{-t^3}{(1 - t)^3(1 - t^2)}.$$

The numerator of $P_Q(t)$ is $-t^3$ and we have $-t^3 \equiv 1 \pmod{(1 + t)}$. Thus

$$P_X(t) = \frac{1 + t^2 + t^3 + t^5}{(1 - t)^4} + \frac{-t^3}{(1 - t)^3(1 - t^2)} = \frac{1 - t^7}{(1 - t)^4(1 - t^2)}.$$

Remark 2.9 The Hilbert series of an isolated orbifold X , appearing as a weighted complete intersection in some format $w\Sigma(\mu, u)$ or cone over $w\Sigma(\mu, u)$, can be computed from the Hilbert series of the $w\Sigma$. In fact, the Hilbert numerator $H(t)$ does not change and the denominator corresponds to the weights of the weighted projective space containing X .

3 Algorithm to compute isolated orbifolds

Let $w\Sigma(\mu, u) \hookrightarrow \mathbb{P}[w_i]$ be the format of dimension d , corresponding to the fixed parameters μ and u . We aim to compute all possible candidate families of n -folds with isolated orbifold points, whose general member is a weighted complete intersection of $w\Sigma(\mu, u)$ or of projective cone(s) over it, denoted by $\mathcal{C}^a w\Sigma$, having fixed canonical weight k . We present an algorithmic approach to compute lists of such n -folds with isolated quotient singularities as a weighted complete intersection in $w\Sigma(\mu, u)$ such that $\mathcal{O}(K_X) = \mathcal{O}(kD)$ for $n \geq 2$, where D is an ample \mathbb{Q} -Cartier divisor.

3.1 The algorithm

Step 1: Compute Hilbert series and canonical class of $w\Sigma$. We start with a fixed weighted flag variety $w\Sigma(\mu, u)$ and compute its Hilbert series $P_{w\Sigma}(t)$. Each choice of input parameters μ and u leads to a codimension e embedding

$$w\Sigma^d(\mu, u) \hookrightarrow \mathbb{P}^m[w_0, \dots, w_m],$$

where $e = m - d$. We choose the parameters μ and u such that $w_i > 0$, for all $i = 0, \dots, m$. The Hilbert series of $w\Sigma$ has the compact form

$$P_{w\Sigma}(t) = \frac{H(t)}{\prod_{i=0}^m (1 - t^{w_i})}. \quad (3.1)$$

If $w\Sigma$ is well-formed, the canonical divisor class $K_{w\Sigma}$ of $w\Sigma$ is given by,

$$\omega_{w\Sigma} = \mathcal{O}_{w\Sigma} \left(q - \sum_{i=0}^m w_i \right) = \mathcal{O}_{w\Sigma}(-p).$$

Since the flag varieties and therefore weighted flag varieties are Fano varieties, $w\Sigma$ is an anti-canonically polarized variety: $\omega_{w\Sigma} = \mathcal{O}_{w\Sigma}(-p)$, where p is an integer: a multiple of u if the Lie group corresponding to $w\Sigma$ is simple.

Step 2: Find all possible embeddings of n -folds X with $\omega_X = \mathcal{O}(k)$. Given the embedding

$$w\Sigma^d \hookrightarrow \mathbb{P}[w_0, \dots, w_m],$$

we find all possible n -folds X as weighted complete intersections inside $w\Sigma$ such that $\omega_X = \mathcal{O}_X(k)$. We intersect $w\Sigma$ with generic hypersurfaces of degrees equal to some weights w_i of the weighted projective space $\mathbb{P}[w_0, \dots, w_m]$:

$$X = w\Sigma \cap (w_{i_1}) \cap \dots \cap (w_{i_l}) \hookrightarrow \mathbb{P}[w_{i_0}, \dots, w_{i_s}]$$

where $s = m - l$. We choose l quasilinear forms (f_j) of degree w_{i_j} , such that $\dim(X) =$

$$d - l = n \text{ and } -p + \sum_{j=1}^l w_{i_j} = k.$$

More generally we can take complete intersections inside projective cones over $\mathcal{C}^a w\Sigma$; we can add some more variables of degree w_i to the coordinate ring which are not involved in any defining relation of $w\Sigma$ and construct X as its quasilinear section with $\omega_X = \mathcal{O}_X(k)$. These newly added variables will be involved in the defining equations of X when we replace some of the variables of $\mathbb{P}^m[w_i]$ with the homogeneous forms (f_j) . The Hilbert numerator $H(t)$ of the Hilbert series of $\mathcal{C}^a w\Sigma$ is the same as that of $w\Sigma$ but we need

to multiply the denominator by $\prod_{i=1}^a (1 - t^{w_i})$, where a is the number of projective cones

$w\Sigma(\mu, u)$. Thus taking each projective cone of degree w_i adds $-w_i$ to the canonical class $K_{w\Sigma}$. This process gives more choices of taking quasilinear sections of an appropriate degree.

Step 3: Compute Hilbert series and the initial term of the n -fold X . We choose one of the n -folds $X \hookrightarrow \mathbb{P}^s[w_{i_0}, \dots, w_{i_s}]$ from the list computed at Step 2, to calculate its Hilbert series $P_X(t)$ and the corresponding initial term $P_I(t)$ as given by (2.3). In fact the Hilbert numerator of the Hilbert series $P_X(t)$ of X will be the same as in (3.1) and the denominator changes to $\prod_{j=0}^s (1 - t^{w_{i_j}})$. The initial term $P_I(t)$, given by (2.4), of the

Hilbert series can be computed from $P_X(t)$ by using first $\lfloor \frac{c}{2} \rfloor + 1$ terms of $P_X(t)$, where $c = k + n + 1$ is the coindex of X .

Step 4: Compute the isolated orbifold loci of $w\mathbb{P}^s$. As we are only interested in computing quasi-smooth candidate orbifolds, the singularities of X are induced by the weights of the weighted projective space $\mathbb{P}^s[w_{i_j}]$. Therefore we find all possible contributions to the basket \mathcal{B} of n -fold isolated cyclic quotient singularities coming from the weights of $\mathbb{P}^s[w_{i_j}]$. Each quotient singularity will be of type $\frac{1}{r_i}(a_{i_1}, \dots, a_{i_n})$ such that

$$\gcd(r_i, a_{i_j}) = 1 \text{ and } \sum_{j=1}^n a_{i_j} + k \equiv 0 \pmod{r_i}.$$

Each n -tuple of integers $(a_{i_1}, \dots, a_{i_n})$ is a sublist of the s -tuple $(w_{i_0}, \dots, w_{i_s})$ and further we require $r_i > 1$ and $r_i \in S \cup G$: $S = \{w_{i_0}, \dots, w_{i_s}\}$ and $G = \{\gcd(S') : S' \subset S\}$.

Step 5: Compute the contributions of orbifold points to the $P_X(t)$. For each isolated orbifold point $Q_i := \left(\frac{1}{r_i}(a_{i_1}, \dots, a_{i_n})\right)$ of the basket \mathcal{B} , we compute its contribution $P_{Q_i}(t)$ to the Hilbert series $P_X(t)$; given by the equation (2.5). We compute the list of all possible baskets coming from the weights of $w\mathbb{P}^{s-1}$, which may contribute the Hilbert series of X . The formula (2.3) can be written as

$$P_X(t) - P_I(t) = \sum_{Q_i \in \mathcal{B}} m_i P_{Q_i}(t); \quad (3.2)$$

Here m_i represents the multiplicities of the isolated orbifold points Q_i , which remains the only unknown in the equation (3.2). It is clear from the equation (3.2) that we need to solve a linear algebra problem

$$[\underline{P}(t)] = [\underline{M}][P_{Q_i}(t)]_{ij}, \quad (3.3)$$

where $P(t) = P_X(t) - P_I(t)$.

Step 6: Examine the candidate n -fold X . The last step is to compute the coefficients m_i appearing in (3.2). If $m_i \geq 0$ and $m_i \in \mathbb{Z}$ for all i , then $(X, \mathcal{O}(k))$ is a suitable candidate for a projectively Gorenstein n -fold with isolated quotient singularities and we restart from step 3 by picking another model of the n -fold computed at step 2. The actual basket of singularities of the candidate variety X will consists of $Q_i \in \mathcal{B}$ such that $m_i > 0$.

Repeat: Step 3–6 We repeat steps 3–6 for all possible n -fold embeddings with $K_X = \mathcal{O}_X(k)$, computed at Step 2.

Remark 3.2 The given algorithm essentially describes the process of finding a list of all possible families of isolated orbifolds with fixed canonical weight k whose general member may be realized as a weighted complete intersection in some prescribed format $w\Sigma(\mu, u)$. But the search for candidate varieties inside the weighted flag variety $w\Sigma$ is an infinite search, as there is no bound on the values of input parameters μ and u . In principle, the algorithm does not give the complete classification of certain type of isolated orbifolds in a given weighted flag variety but essentially computes the complete list up to the certain value of the adjunction number of the Hilbert series, which depend on the values of input parameters μ and u . We search for the candidate varieties until we stop getting new examples or the computer search becomes unreasonably slow due to higher degree weights.

3.3 Implementation of the algorithm

The following remarks describe the implementation of our algorithm in detail.

1. In practice, we search for the candidate orbifolds in some chosen weighted flag variety $w\Sigma$ by running a code for different values of input parameters μ and u . The different values of parameters μ and u may lead to the same set of weights on the embedding $w\Sigma \hookrightarrow w\mathbb{P}V_\lambda$. For instance the two choices of parameters, $\mu_1 = (1, -1)$ and $\mu_2 = (-1, 2)$ with $u = 3$, give the same embedding of the weighted G_2 variety in $\mathbb{P}^{13}[1, 2^4, 3^4, 4^4, 5]$, see Section 4.

To avoid repetition in the computer search, we perform the computer search with pre-determined lists of input parameters μ and u , so that only the distinct embeddings are searched. This allows us to check for candidate orbifolds in distinct embeddings of $w\Sigma$ in some weighted projective space. One can use a simple minded program to compute such lists of input parameters in any computer algebra system. This can essentially be stated as the Step 0 of the search process.

2. The choice of weights on $w\Sigma$, consequently on X , is determined by the values of the parameters μ and u . We arrange the inputs and run the code, in the order of increasing the sum of the weights in $\mathbb{P}[w_0, \dots, w_m]$, i.e. the sum of the weights on the $(i+1)$ -st input is greater or equal than the sum on the i -th input. This is equivalent to ordering the inputs in order of increasing the adjunction number (the degree of the Hilbert numerator $H(t)$) of $w\Sigma$. If $w\Sigma$ corresponds to a simple Lie group G then a bound on u automatically bounds the parameter μ .
3. At Step 2 of the algorithm we compute all possible n -folds X as weighted complete intersections inside $w\Sigma$ or of projective cone(s) over $w\Sigma$ such that $K_X = \mathcal{O}_X(k)$. Since the adjunction number q remains unchanged through the process of taking cone(s) and quasilinear sections, there are only finitely many choices of weights w_i for the embedding

$$X \hookrightarrow \mathbb{P}[w_0, \dots, w_s] \text{ such that } q - \sum_{i=0}^s w_i = k.$$

This essentially makes the process of taking projective cones to be a finite one, which can easily be handled by using simple algorithms.

The geometry behind the construction of orbifolds imposes further conditions on the choices of degree w_i of projective cones and quasilinear sections; which makes the implementation of the algorithm strikingly faster. The degree of the projective cones is bounded by $w_{\max} - 1$. As if the degree of a cone is greater or equal than the maximum weight w_{\max} of ambient space containing $w\Sigma$, the newly introduced variable will not appear in any of the defining equations of $w\Sigma$ and its weighted complete intersections X . Thus a cone of degree greater or equal than w_{\max} will not contribute to the orbifold part of $P_X(t)$; bounding the degree of the projective cone by $w_{\max} - 1$.

Further, the degree of the forms intersecting with $w\Sigma(\mu, u) \subset \mathbb{P}^m[w_i]$ must be equal to one of the ambient weights of the original space containing $w\Sigma(\mu, u)$. Otherwise the process of projective cones will become redundant, in the frame work our construction. The number of projective cones must always be less than or equal to s ; the dimension of ambient weighted projective space containing X .

4. Let

$$\mathcal{B} = \left\{ m_i \times \frac{1}{r_i} (a_{i_1}, a_{i_2}, \dots, a_{i_n}) : 1 \leq i \leq j \right\},$$

be the the basket of isolated orbifold points of $X \hookrightarrow w\mathbb{P}^s$; where m_i represents the multiplicity of the singular point of type $\frac{1}{r_i} (a_{i_1}, a_{i_2}, \dots, a_{i_n})$. Then we define the integer j to be the length of the basket \mathcal{B} . It is evident that $1 \leq j \leq b \leq s$, where b is the number of non-trivial weights of $w\mathbb{P}^s$.

Step 4 of the algorithm computes all possible n -fold distinct isolated orbifold points of the weighted projective $\mathbb{P}[w_0, \dots, w_s]$ which may potentially lie on the candidate orbifold X , contributing to the orbifold part of the $P_X(t)$. The total number B of such orbifold points is usually larger than the actual admissible length b of the basket \mathcal{B} on X , except when the weights of $w\mathbb{P}^s$ are relatively small. In implementing the algorithm, we find the set \mathbb{B} of all the admissible baskets on X ; the length of the baskets range from length 1 up to minimum(b, B). We search for the candidate orbifolds by running the code through all the elements of \mathbb{B} : solving equation (3.2) for each basket \mathcal{B} in \mathbb{B} .

5. Step 6 of the algorithm checks for the solutions m_i to the equation (3.2) for the given basket \mathcal{B} of orbifold points. In certain cases –there may be a kernel of the singular strata– there may exist a collection of orbifold points $\{Q_i\}$ in \mathcal{B} such that $\sum P_{Q_i}(t) = 0$. For example, if $Q_1 = \frac{1}{5}(3, 3, 4)$ and $Q_2 = \frac{1}{5}(1, 2, 2)$ are two orbifold points with $k = 0$ then the corresponding orbifold terms $P_{Q_1}(t)$ and $P_{Q_2}(t)$ are linearly dependent:

$$P_{Q_1}(t) = \frac{t^3 - t^4 + t^5}{(1-t)^3(1-t^5)} = -P_{Q_2}(t).$$

In the implementation of the algorithm, we calculate all possible such combinations of the singular strata of the ambient weighted projective space $w\mathbb{P}^s$ containing the candidate orbifold X . Though from the routine computer search, we can only conclude that the candidate orbifold X either contains some combination of those points with each point having equal multiplicity or contain none of them. But one can precisely answer this question by explicitly computing the orbifold loci of X by using the equations.

6. The equation (3.3) leads to some matrix equation once it is evaluated on the the appropriate j integers, where j is the length of the corresponding basket. The solution of the corresponding matrix represents the multiplicities of the orbifold points of X . For a given isolated orbifold (X, D) the polynomial equation of type (3.2), obtained after appropriately clearing the denominators appearing in $P_I(t)$ and $P_{Q_i}(t)$'s, holds for every value of t . Therefore the corresponding matrix equation (3.3) will always have a unique solution.

We are not given with an isolated orbifold; instead we are rather searching for the one by using the numerics of the Hilbert series and the ambient weighted flag variety. But if the numerics of the singularities and Hilbert series correspond to some isolated orbifold in the given format then the linear system will have a unique solution: which are exactly the cases of our interest. Thus the list we obtain will simply be the over list of the actual number of orbifolds in the given $w\Sigma(\mu, u)$.

4 Applications in 3-fold case

In this section, we apply the algorithm by using a MAGMA code, given in appendix A, to compute lists of candidate 3-folds with k equal to $-1, 0$, and 1 in two types of weighted flag varieties having embedding in codimension 3 and 8. More precisely, we compute lists of Fano 3-folds with isolated log-terminal quotient singularities, Calabi–Yau 3-folds with isolated canonical quotient singularities and canonical 3-folds with terminal quotient singularities. We explicitly construct 5 new families of log-terminal \mathbb{Q} -Fano 3-folds as weighted completed intersection of codimension 8 weighted G_2 variety.

4.1 Weighted G_2 variety: codimension 8

We briefly review the construction of the codimension eight weighted flag variety; a weighted homogeneous variety for the simple Lie group G_2 . The more detailed treatment can be found in [QS11].

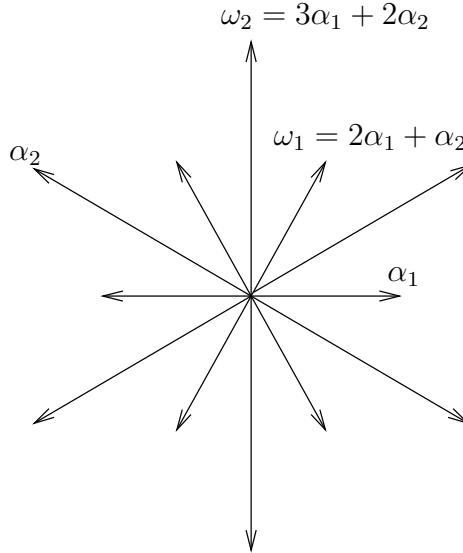


Figure 1: Root System of G_2

Let $\alpha_1, \alpha_2 \in \Lambda_W$ be the pair of simple roots of the root system ∇ of the simple Lie group G_2 . We take α_1 to be the short simple root and α_2 the long one; see Figure 1. The fundamental weights are $\omega_1 = 2\alpha_1 + \alpha_2$ and $\omega_2 = 3\alpha_1 + 2\alpha_2$. The sum of the fundamental weights; also known as Weyl vector ρ is given $\rho = 5\alpha_1 + 3\alpha_2$. The cone of dominant weights is spanned by ω_1 and ω_2 . Then the G_2 -representation with highest weight $\lambda = \omega_2 = 3\alpha_1 + 2\alpha_2$ is 14 dimensional. The corresponding homogeneous variety $\Sigma \subset \mathbb{P}V_\lambda$ is five dimensional, so we have a of codimension 8 embedding $\Sigma^5 \hookrightarrow \mathbb{P}^{13}$. Let $\{\beta_1, \beta_2\}$ be the basis of the lattice Λ_W^* ; dual to $\{\alpha_1, \alpha_2\}$.

The weighted version can be constructed by taking $\mu = a\beta_1 + b\beta_2 \in \Lambda_W^*$ and $u \in \mathbb{Z}^+$; following Section 2.3. The defining equations of the weighted G_2 variety $w\Sigma$ can be calculated by using the decomposition of the second symmetric power $S^2(V_\lambda^*)$ of the dual representation of V_λ , as a module over the Lie algebra \mathfrak{g}_2 : see [QS11] for the details. The 28 quadrics cut out the defining locus of $w\Sigma$, explicitly given in [QS11, Appendix A].

The weights of the ambient weighted projective space $w\mathbb{P}^{13}$ are parameterized by an integer value vector $\mu = (a, b)$ and u ; given by

$$[\pm a + u, \pm b + u, \pm(a + b) + u, \pm(2a + b) + u, \pm(3a + b) + u, \pm(3a + 2b) + u, u, u]. \quad (4.1)$$

As we need all the weights to be positive; there are finite choices of the parameters μ for each positive integer u . The adjunction number q , the degree of the Hilbert numerator $H(t)$, is $11u$. The sum of the weights is $14u$; if $w\Sigma$ is well-formed then the canonical divisor class is $K_{w\Sigma} = \mathcal{O}_{w\Sigma}(-3u)$. We search for the examples in order of increasing the sum of the weights $\sum w_i = 14u$ on $w\mathbb{P}V_\lambda$, which corresponds to the increase in u and consequently in order of increasing the adjunction number $q = 11u$.

4.2 Examples and lists of isolated orbifolds in codimension 8

In this section we prove the existence of some families of isolated log-terminal \mathbb{Q} -Fano 3-folds whose general member can be embedded in a codimension 8 weighted G_2 variety. We also present the list of examples obtained for the Calabi–Yau 3-folds and canonical 3-folds in the codimension 8 weighted G_2 variety.

Theorem 4.1 Let $w\Sigma$ be the codimension eight weighted G_2 -variety. Then there exist 6 families of isolated log-terminal \mathbb{Q} -Fano 3-folds whose general member is a weighted complete intersection in $w\Sigma$ or some projective cone(s) over $w\Sigma$, given by the Table 1.

Table 1 Log-terminal \mathbb{Q} -Fano 3-folds in weighted G_2 variety, (μ, u) -Input parameters, Weights–weights of weighted projective space containing X , $(-K_X)^3$ -Degree of X , Basket–orbifold points of X , and BK–Basket kernel for X .

$(\mu : u)$	Weights	$(-K_X)^3$	Basket	BK
(0,0:1)	$\mathbb{P}[1^{12}]$	18		N
(-1,1:3)	$\mathbb{P}[1, 2^4, 3^4, 4^2, 5]$	$\frac{9}{10}$	$9 \times \frac{1}{2}(1, 1, 1), \frac{1}{5}(3, 4, 4)$	Y
(-1,1:4)	$\mathbb{P}[2, 3^4, 4^3, 5^3]$	$\frac{1}{5}$	$2 \times \frac{1}{2}(1, 1, 1), 6 \times \frac{1}{3}(1, 1, 2), 3 \times \frac{1}{5}(3, 4, 4)$	N
(-2,3:4)	$\mathbb{P}[1^2, 2, 3^2, 4^3, 5^2, 6, 7]$	$\frac{9}{14}$	$2 \times \frac{1}{4}(1, 1, 3), \frac{1}{7}(4, 5, 6)$	N
(-4,6:7)	$\mathbb{P}[1^2, 3, 5^2, 7^3, 9^2, 11, 13]$	$\frac{18}{91}$	$2 \times \frac{1}{7}(1, 2, 5), \frac{1}{13}(7, 9, 11)$	N
(-3,4:7)	$\mathbb{P}[2, 3, 4, 5, 6^2, 7^2, 8, 9, 10, 11]$	$\frac{4}{65}$	$7 \times \frac{1}{2}(1, 1, 1), 3 \times \frac{1}{3}(1, 1, 2), \frac{1}{11}(6, 7, 10)$	Y

Proof The proof is essentially to construct all the candidate examples listed in Table 1, one after another. We start with the first entry listed in Table 1, then we have the following data to begin with:

- Input: $\mu = (-1, 1)$, $u = 3$
- Variety and weights: $w\Sigma \subset \mathbb{P}^{13}[1, 2^4, 3^4, 4^4, 5]$

- Canonical class: $K_{w\Sigma} = \mathcal{O}(-9)$, as we can check that $w\Sigma$ is well-formed.
- Hilbert Numerator: $1 - 3t^4 - 6t^5 - 8t^6 + 6t^7 + 21t^8 + \dots + 6t^{26} - 8t^{27} - 6t^{28} - 3t^{29} + t^{33}$

We take a threefold quasilinear section, by intersecting $w\Sigma$ with two general forms of degree four

$$X = w\Sigma \cap (4)^2 \hookrightarrow \mathbb{P}^{11}[1, 2^4, 3^4, 4^2, 5] \text{ with } K_X = \mathcal{O}(-1).$$

We check all the singular strata of X by using the equations given in [QS11, Appendix A].

1/5 singularities: Since we are not taking any degree five sections of $w\Sigma$ and the variable of weight 5 does not appear as monomial of degree 2 in the defining equations of $w\Sigma$, X contains this point. By using the implicit function theorem we find the local transverse parameters near this point to be of degree 4, 4 and 3. Therefore X has a singular point of type $\frac{1}{5}(3, 4, 4)$.

1/4 singularities: X does not contain this singular point.

1/3 singularities: X also avoids singularities of type $\frac{1}{3}$.

1/2 singularities X contains 9 singular points of type $\frac{1}{2}(1, 1, 1)$.

Thus $(X, -K_X)$ is a log-terminal \mathbb{Q} -Fano threefold with $(-K_X)^3 = \frac{9}{10}$ and with $9 \times \frac{1}{2}(1, 1, 1), \frac{1}{5}(3, 4, 4)$ singular points.

The existence of the remaining log-terminal \mathbb{Q} -Fano 3-folds appearing in Table 1, has been established by using explicit equations and a similar analysis of the singular strata. \square

Remark 4.3 If X is a terminal \mathbb{Q} -Fano 3-fold appearing as a quasilinear section of some weighted G_2 variety with parameters $(\mu : u)$, then the adjunction number of X will be $11u$. The list of all possible codimension 8 terminal \mathbb{Q} -Fano 3-fold, with their basket of singularities, adjunction number, degree etc; is available on the graded ring database page [BK]. The highest adjunction number for the codimension 8 terminal \mathbb{Q} -Fano 3-folds which is an integer multiple of 11 is 66 in the database, so a terminal \mathbb{Q} -Fano 3-fold in the weighted G_2 variety can exist only for $u \leq 6$. But the computer search does not produce even a candidate terminal \mathbb{Q} -Fano 3-fold for $1 \leq u \leq 6$, leading to the following corollary.

Corollary 4.4 There does not exist a terminal \mathbb{Q} -Fano 3-fold in codimension 8 which can be realized as weighted complete intersection in codimension 8 weighted G_2 variety.

Remark 4.5 Table 1 does not represent a complete classification of isolated log-terminal \mathbb{Q} -Fano 3-folds in the codimension 8 weighted G_2 variety. Certainly, it is a sublist of the complete classification of such 3-folds in the given weighted G_2 variety, as the computer search has been completely performed only up to $u = 7$. One should expect more examples for the higher values of u . Similarly Table 2 and Table 3 do not represent the full classification of Canonical and Calabi–Yau 3-folds respectively. On the other hand the computer routine returns an over list of numerical candidates, up to certain values of the adjunction number q . In the case of log-terminal \mathbb{Q} -Fano 3-folds, we get a list of 33 suggested models of candidate orbifolds up to $q = 77$ but only 6 of them exist as an actually variety with the suggested invariants and singularities. We include the smooth Fano 3-fold of genus 10 in the list for the sake of completeness as it already appeared in [QS11]. The existence of these candidates is established by checking through the equations of each of these 33 numerical candidates. The checking is done partly by hand and partly by the computer algebra, as the number of defining equations is substantially large. The

following example represents a suggested candidate orbifold which does not exists as an actual variety with the suggested numerical data.

Example 4.6 If we take the $\mu = (-3, 4)$ and $u = 6$, then we get the embedding

$$w\Sigma(\mu, u) \hookrightarrow \mathbb{P} [1, 2, 3, 4, 5^2, 6^2, 7^2, 8, 9, 10, 11] ,$$

with $K_{w\Sigma} = \mathcal{O}(-18)$. Then a 3-fold quasilinear section

$$X = w\Sigma \cap (10) \cap (7) \hookrightarrow \mathbb{P} [1, 2, 3, 4, 5^2, 6^2, 7, 8, 9, 11]$$

has $K_X = \mathcal{O}_X(-18+10+7) = \mathcal{O}_X(-1)$. The computer search suggests a numerical candidate log-terminal \mathbb{Q} -Fano 3-fold with the basket of singularities $\mathcal{B} = \{\frac{1}{11}(6, 8, 9), \frac{1}{5}(1, 1, 4), 2 \times \frac{1}{2}(1, 1, 1)\}$ and $(-K_X)^3 = \frac{9}{55}$. By checking through the equations of X , induced from the equations of $w\Sigma$, the orbifold points of type $\frac{1}{11}(6, 8, 9)$ and $\frac{1}{5}(1, 1, 4)$ lie on X . But the fix locus under the action of cyclic group \mathbb{Z}_2 is an empty subscheme of X ; the 2 points of type $\frac{1}{2}(1, 1, 1)$ do not actually lie on X . Thus the suggested candidate 3-fold model can not be realised as a quasilinear section of weighted G_2 variety with the given numerical data.

Table 2 Candidate canonical 3-folds in the codimension 8 weighted G_2 variety

$(\mu : u)$	Weights	$(K_X)^3$	Basket
(0,0:2)	$\mathbb{P} [1^3, 2^9]$	9	$18 \times \frac{1}{2}(1, 1, 1)$
(-1,1:3)	$\mathbb{P} [1^2, 2^4, 3^3, 4^2, 5]$	$\frac{27}{10}$	$9 \times \frac{1}{2}(1, 1, 1), \frac{1}{5}(1, 4, 4)$
	$\mathbb{P} [1, 2^5, 3^4, 4^3]$	$\frac{9}{5}$	$18 \times \frac{1}{2}(1, 1, 1), \frac{1}{5}(2, 3, 4)$
	$\mathbb{P} [1, 2^4, 3^5, 4^2]$	$\frac{9}{5}$	$9 \times \frac{1}{2}(1, 1, 1), 6 \times \frac{1}{3}(1, 2, 2)$
(-2,3:4)	$\mathbb{P} [1^2, 2^2, 3^2, 4^2, 5^2, 6, 7]$	$\frac{9}{7}$	$2 \times \frac{1}{2}(1, 1, 1), \frac{1}{7}(2, 5, 6)$
	$\mathbb{P} [1, 2^2, 3^3, 4^2, 5^3, 6]$	$\frac{3}{5}$	$2 \times \frac{1}{2}(1, 1, 1), 3 \times \frac{1}{3}(1, 2, 2), 2 \times \frac{1}{5}(2, 3, 4)$
(-1,1:4)	$\mathbb{P} [1, 2, 3^3, 4^4, 5^3]$	$\frac{3}{5}$	$2 \times \frac{1}{2}(1, 1, 1), 3 \times \frac{1}{5}(1, 4, 4)$
	$\mathbb{P} [2^2, 3^4, 4^3, 5^3]$	$\frac{2}{5}$	$4 \times \frac{1}{2}(1, 1, 1), 6 \times \frac{1}{3}(1, 2, 2), 3 \times \frac{1}{5}(1, 4, 4)$
(-2,3:5)	$\mathbb{P} [2^2, 3^2, 4^2, 5^3, 6, 7, 8]$	$\frac{9}{40}$	$11 \times \frac{1}{2}(1, 1, 1), 2 \times \frac{1}{5}(1, 4, 4), \frac{1}{8}(3, 5, 7)$
	$\mathbb{P} [2, 3^3, 4^2, 5^3, 6, 7^2]$	$\frac{6}{35}$	$6 \times \frac{1}{3}(1, 2, 2), 2 \times \frac{1}{5}(1, 4, 4), \frac{1}{7}(3, 4, 6)$
(-2,3:6)	$\mathbb{P} [2, 3^2, 4, 5^2, 6^2, 7^2, 8, 9]$	$\frac{1}{9}$	$2 \times \frac{1}{2}(1, 1, 1), 7 \times \frac{1}{3}(1, 2, 2), \frac{1}{9}(2, 7, 8)$
(-3,4:7)	$\mathbb{P} [2, 3, 4^2, 5, 6, 7^2, 8, 9, 10, 11]$	$\frac{3}{44}$	$10 \times \frac{1}{2}(1, 1, 1), \frac{1}{4}(1, 3, 3), \frac{1}{11}(4, 7, 10)$

Remark 4.7 Since the set of orbifold contributions for the canonical terminal points $\frac{1}{r}(-1, a, -a)$ on 3-folds with $k = 1$ are linearly independent [IF89], we do not include the basket kernel column in Table 2.

Table 3 Candidate canonical Calabi–Yau 3-folds in the codimension 8 weighted G_2 variety

$(\mu : u)$	Weights	D^3	Basket	Ker
$(-1, 1:3)$	$\mathbb{P}[1, 2^4, 3^5, 4, 5]$	$\frac{6}{5}$	$6 \times \frac{1}{3}(1, 1, 2), \frac{1}{5}(3, 3, 4)$	Y
$(-2, 3:4)$	$\mathbb{P}[1^2, 2, 3^3, 4^2, 5^2, 6, 7]$	$\frac{6}{7}$	$3 \times \frac{1}{3}(1, 1, 1), \frac{1}{7}(3, 5, 6)$	
$(-2, 3:5)$	$\mathbb{P}[2^2, 3^2, 4, 5^3, 6^2, 7^2]$	$\frac{6}{35}$	$3 \times \frac{1}{3}(2, 2, 2), 2 \times \frac{1}{5}(1, 2, 2), \frac{1}{7}(2, 6, 6)$	
$(-3, 4:7)$	$\mathbb{P}[2, 3, 4, 5^2, 6, 7^2, 8, 9, 10, 11]$	$\frac{3}{55}$	$2 \times \frac{1}{5}(1, 2, 2), \frac{1}{11}(5, 7, 10)$	
	$\mathbb{P}[3, 4, 5^2, 6^2, 7^3, 8, 9, 10]$	$\frac{1}{35}$	$7 \times \frac{1}{2}(1, 1, 1), 3 \times \frac{1}{3}(1, 1, 2), \frac{1}{11}(6, 7, 10)$	

4.8 Computations of other known lists

There are some famous lists of 3-dimensional orbifolds, for example 95 \mathbb{Q} -Fano 3-fold hypersurfaces in weighted projective spaces [IF00] or 69 families of codimension 3 \mathbb{Q} -fano 3-folds in the weighted Grassmannian $w\text{Gr}(2, 5)$ format etc. The summary of such lists and corresponding references can be found in Table 1 of [BKZ]; where the results were obtained by using a different approach than ours. Since weighted projective spaces are a particular type of weighted flag varieties; those lists of orbifolds can be recovered by making slight modifications to our computer routine, except the lists of codimension 1 Calabi–Yau 3-folds of [KS00]. Theoretically, we will eventually obtain the full lists of these 3-folds as well but the computer search gets hopelessly slow once the weights of the ambient space get larger.

In theory, the algorithm can be used to find lists of orbifolds inside any ambient weighted projective variety with a computable canonical divisor class and Hilbert series. As a description, we recover lists of canonical, Calabi–Yau and log-terminal \mathbb{Q} -Fano 3-folds (including the terminal \mathbb{Q} -Fano 3-folds) inside weighted Grassmannian $w\text{Gr}(2, 5)$ format and present the results in table 4.

Table 4 presents the summary of the results obtained in two cases: codimension 8 weighted G_2 variety and codimension 3 weighted Grassmannian $\text{Gr}(2, 5)$. In each case the results are searched up to the adjunction number q_{\max} . The number q_{res} represents the adjunction number for which the last numerical 3-fold was found. The column $\#w\Sigma_{\text{dis}}$ gives the number of distinct embeddings searched in the given format, and $\#w\Sigma_{\text{res}}$ is the number of the embedding where the last result was found. The column $\#\text{output}$ represents the number of suggested candidate 3-folds and $\#\text{result}$ gives the number of plausible candidate 3-folds in the given weighted flag variety.

In the case of $w\text{Gr}(2, 5)$, we recover the list of 18 canonical for 3-folds computed in [BKZ] for $k = 1$. The list of famous 69 families of \mathbb{Q} -Fano 3-folds in codimension 3 is obtained as a sublist of 403 numerical examples of log-terminal \mathbb{Q} -Fano 3-folds computed for $k = -1$. In the case of $k = 0$ the number of Calabi–Yau 3-folds obtained up to adjunction number 71 is an over list of the 187 such 3-folds appearing on the graded ring database page [BK], computed by using a different approach than ours.

For the codimension 8 weighted G_2 variety the adjunction number increases quite rapidly; leading to fewer cases of distinct embeddings. For $k = 1$ we obtained a list of 14 numerical candidates with 12 plausible examples and the case of $k = 0$ gives 13 candidate families of

Table 4 The summary of results showing the number of families of log-terminal \mathbb{Q} -Fano 3-folds, Calabi–Yau 3-folds, and canonical 3-folds with isolated orbifold points in two formats: codimension 8 G_2 , and $\text{Gr}(2, 5)$. The column $\#w\Sigma_{\text{dis}}$ gives the number of distinct embeddings searched for examples, $\#w\Sigma_{\text{last}}$ is the last embedding where the example appeared, q_{res} gives the largest adjunction number for which a result was found; q_{max} gives the largest adjunction number searched; $\#outputs$ gives the number of candidates found by the computer; $\#results$ gives the number of candidates after removing the candidates with obvious failure.

Format	codim	k	$\#w\Sigma_{\text{dis}}$	$\#w\Sigma_{\text{res}}$	q_{max}	q_{res}	$\#outputs$	$\#results$
G_2	8	-1	23	19	77	77	32	6
		0	41	27	99	88	12	6
		1	53	19	110	77	14	12
$\text{Gr}(2, 5)$	3	-1	17180	13403	63	63	403(69)	
		0	29941	29165	71	71	221	187
		1	29941	1196	71	35	18	18

Calabi–Yau 3-folds with 6 of them being plausible. For $k = -1$ we get 33 numerical candidates of isolated log-terminal \mathbb{Q} -Fano 3-folds with 6 of them existing as actual varieties; explicitly constructed in Theorem 4.1.

Remark 4.9 There are some other types of weighted flag varieties constructed in [QS11, Qur15] in codimension 6, 7, and 9. The detailed list of isolated orbifolds in those weighted flag varieties will appear elsewhere [BKQ]. In this article, we restrict the attention to description of the algorithm and its applications to sample cases.

A Magma code to compute n -folds

This code consists of the main MAGMA function "Format", which uses some auxiliary functions and some extra data to produce the required lists of examples. The following is the most general form of the implementation of our algorithm. The search process can be significantly fastened by a slight modification in the code in particular cases. For example, in the case of isolated Calabi–Yau 3-folds the index of singularity must be odd, so a minor modification in the function "Porb_Cont" fastens the search significantly.

For the whole calculation we run the following basic commands to start calculations after logging into MAGMA.

```
Q:=Rationals();
R<t>:=PolynomialRing(Q);
K:=FieldOfFractions(R);
S<s>:=PowerSeriesRing(Q,50);
```

The function Qorb calculates the contribution $P_{Q_i}(t)$ of each isolated singular point $\frac{1}{r_i}(a_1, \dots, a_n)$ to the Hilbert series $P_X(t)$ of X . The input to this function are the index of singularity r , the

weights of the local coordinates $LL = [a_1, \dots, a_n]$ and the canonical weight kx of X . This function is the own implementation of their algorithm by the authors of [BRZ13].

```
function Qorb(r,LL,kx)
L := [ Integers() | i : i in LL ];
    if (kx + &+L) mod r ne 0
    then error "Error: Canonical weight not compatible";
    end if;
n := #LL; Pi := &*[ R | 1-t^i : i in LL];
h := Degree(GCD(1-t^r, Pi)); l := Floor((kx+n+1)/2+h);
de := Maximum(0,Ceiling(-l/r)); m := l + de*r;
A := (1-t^r) div (1-t); B := Pi div (1-t)^n;
H,al_throwaway,be:=XGCD(A,t^m*B);
return t^m*be/(H*(1-t)^n*(1-t^r)*t^(de*r));
end function;
```

The function "Init.Term" computes the contribution of the initial term $P_I(t)$ of the Hilbert series $P_X(t)$ of X , as given by (2.3). The input is: hs - the Hilbert series of X , n - dimension of X , and c -coindex of X .

```
function Init_Term(hs,c,n)
co:=Coefficients(S!hs)[1..Floor(c/2)+1];
f:=&+[co[i]*t^(i-1): i in [1..#co]];
pp:=R!(f*(1-t)^(n+1));
    if IsEven(c) eq true then
    return (&+[Coefficient(pp, i )*(t^i+t^(c-i)):i in [0..c div 2-1]]+
    Coefficient(pp,c div 2)*t^(Floor(c/2)))/(1-t)^(n+1);
    else
    return &+[Coefficient(pp,i)*(t^i+t^(c-i)):i in [0..Floor(c/2)]]
    /(1-t)^(n+1);
    end if;
end function;
```

The function "Pos.Wt" computes all possible embeddings of $X \hookrightarrow w\mathbb{P}^{s-1}$, as a quasi-linear section of $w\Sigma \hookrightarrow \mathbb{P}L$ and of all possible projective cone(s) over it, with a desired canonical divisor class. As an input we use the weights of the embedding $w\Sigma \hookrightarrow \mathbb{P}[w_i]$ as a list of integers L and integers s, w , where w is the required sum of the weights on \mathbb{P}^{s-1} . As an output we get lists of integer lists of length s such that their sum is w and corresponding ambient weighted projective space $\mathbb{P}^{s-1}[w_i]$ is well-formed.

```
function Pos_Wt(L,s,w)
PosWt:=[Sort(p): p in RestrictedPartitions(w,s,{1..Max(L)}) | Multiplicity(p,Max(L))
le Multiplicity(L,Max(L)) and (&+[GCD(Remove(p,i)): i in [1..#p]] eq s)];
return PosWt;
end function;
```

The function "Porb.Cont" calculates all the possible singularities coming from the embedding of X . As an input it takes the list of weights of $w\mathbb{P}^{s-1}$, the canonical weight kx and the dimension n of X .

```

function Porb_Cont(weights,kx,n)
LL:=[PowerSequence(Integers())|]; R:=[Integers()|];
for r in weights do
gcds:={GCD(r,s) : s in weights};
weights cat:= [p : p in gcds | p ne 1 and p not in weights];
end for;
set:=SequenceToSet(Sort(weights));
  for r in {w : w in set | w ne 1} do
    rowt:=Sort([a mod r : a in weights | GCD(a,r) eq 1]);
    if (#rowt ge n) then
      N:=SetToSequence(Subsets({1..#rowt},n));
      I:=[SetToSequence(N[i]): i in [1..#N]];
      BI:=[Sort(rowt[I[j]]): j in [1..#N] | (m mod r eq 0) where
m is &+rowt[I[j]]+kx];
      tr:= Setseq(Seqset(BI));
      for L in tr do
        LL:=Append(LL,L);R:=Append(R,r);
      end for;
    end if;
  end for;
return R,LL,weights;
end function;

```

The function "Baskets" computes all possible baskets which may lie on X , induced from the weights of embedding $\mathbb{P}^{s-1}[w_i]$. The input of this function is the output from the function "Porb_Cont". we know that $X \hookrightarrow \mathbb{P}^{s-1}[w_i]$, so the maximum length (defined in the Section 3.3) of the baskets is s .

```

function Baskets(R,LL,weights)
RR:=[PowerSequence(Integers())|];
BB:=[PowerSequence(PowerSequence(Integers()))|];
for s in [1..Min(#R,#weights)] do
  for I in {<R[Sort(SetToSequence(m))],LL[Sort(SetToSequence(m))]> :
m in Subsets({1..#R},s)} do
    if #(SequenceToMultiset(I[1]) meet
SequenceToMultiset(weights)) ge s then
      Append(~RR,I[1]);
      Append(~BB,I[2]);
    end if;
  end for;
end for;
return RR,BB;
end function;

```

The function "Bask_Kernel" computes the kernel of the basket of singularities \mathcal{B} induced from the weights of $\mathbb{P}^{s-1}[w_i]$, i.e. given a set of isolated orbifold points Q_i , it computes all possible

combinations of Q_i such that $\sum P_{Q_i}(t) = 0$. As an input it takes the weights of the ambient space, the output from the function `Baskets`, and the canonical weight kx of X .

```
function Basket_Kernel(weight,RR,BB,kx)
if #RR ge 2 then
  BR:=[PowerSequence(Integers())];BL:=[PowerSequence(PowerSequence(Integers()))];
  for s in [2..Min(#weight,#RR)] do
    for I in {<RR[Sort(SetToSequence(m))],BB[Sort(SetToSequence(m))]> :
      m in Subsets({1..#RR},s)} do
      if (#(SequenceToMultiset(I[1]) meet SequenceToMultiset(weight)) eq s)
        and (&+[Qorb(I[1][j],I[2][j],kx): j in [1..#I[1]]] eq 0) then
        Append(~BR,I[1]); Append(~BL,I[2]);
      end if;
    end for;
  end for;
end if;
return BR,BL;
end function;
```

The procedure "`Format`" is the main function utilizing the rest of the functions in combination to search for a suitable candidate orbifold (X, D) . As an input it takes the following data of $w\Sigma(\mu, u)$ and X .

- `num`- The Hilbert numerator of $w\Sigma(\mu, u)$ and hence of X
- `wtsigma`- The weights of the embedding $w\Sigma \hookrightarrow w\mathbb{P}$, as a list of integers
- `n`- The dimension of desired candidate orbifold (X, D)
- `kx`- is the canonical weights of X
- `s`- is the number coming from $X \hookrightarrow w\mathbb{P}^{s-1}$

This function checks for plausible candidate isolated orbifolds in a given weighted flag variety $w\Sigma(\mu, u)$ and all possible projective cones over it. If such a candidate is found it will output the candidate with its degree, singularities, weights of the embedding and Hilbert numerator, otherwise only the Hilbert numerator is returned to show the completion of the process.

```
function Format(num,kx,s,wtsigma,n)
for weight in Pos_Wt(wtsigma,s,Degree(num) - kx) do
den:=%*[1 - t^n : n in weight];
px:=num / den;
deg:=Evaluate(px * (1 - t)^(n + 1),1);
pini:=Init_Term(px,n + kx + 1,n);
if px eq pini then
printf "Smooth %o-fold with canonical class K_X=0(%o)\n" cat
"Ambient: P%o\nDegree: %o\n",
n,kx,weight,deg;
else
```

```

Ra,LLa,weights:=Porb_Cont(weight,kx,n);
RR,BB:=Baskets(Ra,LLa,weights);
  for i in [j : j in [1..#RR] | #RR[j] ne 0] do
    R:=RR[i];
    B:=BB[i];
    Porb:=[Qorb(R[j],B[j],kx) : j in [1..#R]];
    den:=&*[Denominator(PQ) : PQ in Porb] *Denominator(px) * Denominator(pini);
    PX:=den * px;
    Pini:=den * pini;
    Porb:=[PQ * den : PQ in Porb];
    LHS:=PX - Pini;
    if Min([Degree(LHS) - Degree(PQ) : PQ in Porb]) ge 0 then
      V:=Vector(Integers(),[Evaluate(LHS,k) : k in [2..#R + 1]]);
      A:=Matrix([PowerSequence(Integers())| [Evaluate(PQ,k):k in [2..#R+ 1]]:
        PQ in Porb]);
      ok,sol:=IsConsistent(A,V);
      if ok and Min(Eltseq(sol)) ge 0 and
        (LHS - &+[sol[m]*Porb[m]:m in [1..#R]] eq 0) then
        BR,BL:=Basket_Kernel(weight,Ra,LLa,kx);
        if #BR ge 1 and [&+Sum,&*Prod,&+Ind] notin SS then
          printf "Isolated %o-fold with canonical class K_X=0(%o)\n" cat
            "Ambient: P%o\nSingularities:\n %o x 1/%o\n%o\n" cat
            "Degree:%o\n\n\n\n" cat
            "Basket Kernel:1/%o x %o\n\n",
            n,kx,weight,sol,R,B,deg,BR,BL;
        else
          printf "Isolated %o-fold with canonical class K_X=0(%o)\n" cat
            "Ambient: P%o\nSingularities:\n %o x 1/%o\n%o\n" cat
            "Degree:%o\n\n\n\n",
            n,kx,weight,sol,R,B,deg;
        end if;
      end if;
    end if;
  end for;
end for; return "Hilbert Numerator:" num;
end function;

```

References

- [ABR02] Selma Altınok, Gavin Brown, and Miles Reid, *Fano 3-folds, K3 surfaces and graded rings*, Topology and geometry: commemorating SISTAG, Contemp. Math., vol. 314, Amer. Math. Soc., Providence, RI, 2002, pp. 25–53.
- [AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

- [BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993). MR MR1484478
- [BK] G. Brown and A. M. Kasprzyk, *Graded ring database, available at*, www.grdb.co.uk.
- [BKQ] G.D. Brown, A. M. Kasprzyk, and M. I. Qureshi, *Fano fourfolds in Gorenstein formats*, In preparation.
- [BKR12] Gavin Brown, Michael Kerber, and Miles Reid, *Fano 3-folds in codimension 4, Tom and Jerry. I.*, Compos. Math. **148** (2012), no. 4, 1171–1194.
- [BKZ] G. Brown, A. M. Kasprzyk, and L. Zhu, *Gorenstein formats, canonical and Calabi-Yau threefolds*, arXiv:1409.4644.
- [Bor96] A. Borisov, *Boundedness theorem for Fano log-threefolds*, J. Algebraic Geom. **5** (1996), no. 1, 119–133.
- [BRZ13] A. Buckley, M. Reid, and S. Zhou, *Ice cream and orbifold Riemann–Roch*, Izvestiya: Mathematics **77:3** (2013), 461–486.
- [CR02] A. Corti and M. Reid, *Weighted Grassmannians*, Algebraic geometry (M. C. Beltrametti, F. Catanese, C. Ciliberto, A. Lanteri, and C. Pedrini, eds.), de Gruyter, Berlin, 2002, pp. 141–163.
- [IF89] A. R. Iano-Fletcher, *Inverting reid’s exact plurigenera formula*, Math. Ann. **84** (1989), no. 4, 617–629.
- [IF00] ———, *Working with weighted complete intersections*, Explicit Birational Geometry of 3-folds, vol. 281, London Math. Soc. Lecture Note Ser, CUP, 2000, pp. 101–173.
- [Kaw84] Y. Kawamata, *The cone of curves of algebraic varieties*, Ann. of Math. (2) **119** (1984), no. 3, 603–633.
- [KS00] M. Kreuzer and H. Skarke, *Complete classification of reflexive polyhedra in four dimensions*, Adv. Theo. Math. Phys. **4** (2000), no. 6, 1209–1230.
- [Muk88] S. Mukai, *Curves, K3 surfaces and Fano 3-folds of genus ≤ 10* , Algebraic geometry and commutative algebra, Vol. I, Kinokuniya, 1988, pp. 357–377.
- [Muk89] ———, *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3*, Proc. Nat. Acad. Sci. U.S.A. **86** (1989), no. 9, 3000–3002.
- [QS11] Muhammad Imran Qureshi and B. Szendrői, *Constructing projective varieties in weighted flag varieties*, Bull. Lon. Math. Soc. **43** (2011), no. 2, 786–798.
- [QS12] Muhammad Imran Qureshi and Balázs Szendrői, *Calabi-Yau threefolds in weighted flag varieties*, Adv. High Energy Phys. (2012), Art. ID 547317, 14 pp.
- [Qur15] Muhammad Imran Qureshi, *Constructing projective varieties in weighted flag varieties II*, Math. Proc. Camb. Phil. Soc. **158** (2015), 193–209.

DEPARTMENT OF MATHEMATICS,
LUMS SCHOOL OF SCIENCE AND ENGINEERING,
U-BLOCK, DHA, LAHORE, PAKISTAN
Email address: `i.quareshi@maths.oxon.org`